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Cosmological Evolution of Scale-Invariant Gravity Waves

Kin-Wang Ng and A. D. Speliotopoulos

Institute of Physics, Academia Sinica

Taipei, Taiwan 115, R.O.C.

The evolution of scale-invariant gravity waves from the early universe is analyzed using an equation of state which smoothly interpolates between the radiation dominated era and the present matter dominated era. We find that for large wavenumbers the standard scale-invarant wavefunction for the gravity wave severely *underestimates* the actual size of the gravity wave. Moreover, there is a definite shift in the *phase* of the gravity wave as it crosses the radiation-matter phase transition. The tensor-induced anisotropy of the cosmic microwave background and the present spectral energy density of the gravity wave is then calculated using these results.

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It is well known that a scale-invariant (SI) spectrum for gravity waves (GW) can be produced in an inflationary cosmology [1]. During inflation, fluctuations in the gravitational field will be red-shifted out of the horizon, after which they freeze and remain at a constant amplitude. Much later when they re-enter the horizon during either the radiation-dominated (RD) or matter-dominated (MD) era, they will appear as classical GW. Via the Sachs-Wolfe effect [2], these primordial GW will leave imprint on the cosmic microwave background (CMB) [3]. In fact, recent measurements of the power spectrum extracted from the large-angular-scale anisotropy of CMB [4] suggests a behavior that may be explained by these primordial GW [5,6]. Moreover, the direct detection of GW is one of the main goal of many future experiments. Tracing the precise evolution of these GW is therefore mandatory and important.

In this Letter we shall analyse in detail the time evolution of a SI spectrum of GW in the standard hot big-bang cosmology. This subject has been extensively studied before, but in all previous approaches one either considers only the long wavelength behavior of the GW, and/or a sudden change of equation of state as the universe moves from the RD to MD eras is adopted. Here the novelty is that we shall include a smooth radiation-matter phase transition, which will allow us to obtain specific solutions of the wave equation across the transition. By doing so, we shall find a spectrum for the short-wavelength GW that is radically different from what is currently accepted. Moreover, we shall be able to explicitly determine the phase of the GW. We then apply our result in the calculation of the tensor-induced anisotropy of CMB, and the present spectral energy density of GW. Attempts at incorporating a smooth RD to MD transition in the equation of state has also been made by using transfer function methods [7].

We assume a spatially flat two-component universe containing radiation and dust. The metric that we will use throughout is the flat Robertson-Walker metric: $ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\eta')(d\eta'^2 - d\mathbf{x}^2)$ where $a(\eta')$ and $d\eta' = dt/a(t)$ are the scale factor and conformal time respectively. We will consider a monochromatic plane GW of wavenumber k'. We first define the new variables $\eta \equiv (\sqrt{2}-1)\eta'/\eta'_{eq}$, $k \equiv k'\eta'_{eq}/(\sqrt{2}-1)$. η'_{eq} is the time at which the energy density of radiation is equal to that of matter. The equation of motion of the GW amplitude is then given by [8]

$$\frac{d^2h}{d\eta^2} + \frac{2}{a}\frac{da}{d\eta}\frac{dh}{d\eta} + k^2 h = 0. \tag{1}$$

In a generic inflationary model, a SI spectrum of GW can cover a very wide range of k (for instance, $10^{-5} < k < 10^{23}$) [9]. For these k's the standard boundary conditions for $h(\eta)$ are

$$h(\eta) = 1 \qquad \text{and} \qquad \frac{dh(\eta)}{d\eta} = 0 \qquad \text{as} \qquad \eta \to 0. \tag{2}$$

In the standard solution of a SI spectrum, which we shall call the SI solutions, Eq. (1) is solved in the RD and MD universes seperately [3] giving

$$h_{rm}(\eta) = \begin{cases} j_0(k\eta), & \text{if } \eta \ll \eta_{eq}; \\ 3j_1(k\eta)/(k\eta), & \text{if } \eta \gg \eta_{eq}, \end{cases} \tag{3}$$

where $j_\nu(k\eta)$ and $y_\nu(k\eta)$ (below) are the spherical Bessel functions of order ν of the first and second kind, respectively.

As a first attempt to take into account the radiation-matter phase transition in the analysis of the evolution of GW, one typically makes the sudden approximation

$$\frac{2}{a}\frac{da}{d\eta} = \begin{cases} 2/\eta, & \text{if } \eta < \eta_{eq}; \\ 4/(\eta + \eta_{eq}), & \text{if } \eta > \eta_{eq}. \end{cases} \tag{4}$$

Once again it is straightforward to solve Eq. (1) giving

$$h_s(\eta) = \begin{cases} j_0(k\eta), & \text{if } \eta < \eta_{eq}; \\ (\eta_{eq}/\eta)\left(A_s j_1(k\eta) + B_s y_1(k\eta)\right), & \text{if } \eta > \eta_{eq}. \end{cases} \tag{5}$$

The subscript s denotes the fact that we are using the sudden approximation (4) . Notice that $h_s(\eta)$ satisfies the boundary conditions (2), while the integration constants A_s and B_s are determined by requiring $h(\eta)$ and $dh/d\eta$ be continuous at $\eta = \eta_{eq}$. This gives

$$\begin{aligned} A_s &= 4\left(\sin k\eta_{eq} + \frac{\cos k\eta_{eq}}{2k\eta_{eq}} + \frac{\sin k\eta_{eq}}{4(k\eta_{eq})^2} - \frac{\sin^3 k\eta_{eq}}{2(k\eta_{eq})^2}\right), \\ B_s &= -4\left(\cos k\eta_{eq} - \frac{\sin k\eta_{eq}}{2k\eta_{eq}} - \frac{\cos k\eta_{eq}}{2(k\eta_{eq})^2} + \frac{\cos^3 k\eta_{eq}}{2(k\eta_{eq})^2}\right). \end{aligned} \tag{6}$$

Notice also that $h_s \to 1$ as $k\eta \to 0$, and by this criterion h_s is a scale-invariant solution. Similar solutions can also be found in Ref. [10].

Of course, the evolution of the universe is not nearly this sudden. There is instead a gradual change in the equation of state from the early RD era into the MD universe of the present day with the scale factor

$$a(\eta) = a_{eq}\eta\,(\eta + 2)\ \ . \tag{7}$$

a_{eq} is the scale factor at $\eta = \eta_{eq}$. (Note that $\eta_{eq} \approx 0.41$ and the present time $\eta_0 \approx 76.5$ if we take $a_0/a_{eq} = 6000$.) Then Eq. (1) reduces to

$$\frac{d^2h}{d\eta^2} + \frac{4(\eta+1)}{\eta(\eta+2)}\frac{dh}{d\eta} + k^2h = 0. \tag{8}$$

This equation can be solved exactly in terms of spheroidal wave functions [11,12]. Doing so will not be very illuminating and we shall instead use the WKB approximation to obtain an approximate analytical expression.

First, we define

$$h(\eta) = \frac{y(\eta)}{\eta(\eta+2)}. \tag{9}$$

Then Eq. (8) becomes

$$\frac{d^2y}{d\eta^2} + \left(k^2 - \frac{2}{\eta(\eta+2)}\right)y = 0, \tag{10}$$

which is analogous to the Schröedinger equation for the "wavefunction" of the particle with energy k^2 in a "radial" potential $V(\eta) = 2/\{\eta(\eta+2)\}$ since $\eta \geq 0$. Because the boundary conditions for $h(\eta)$ are given at $\eta = 0$, the problem now reduces to the quantum mechanical problem of a particle with energy k^2 tunneling out of the potential $V(\eta)$.

Notice that in this language the sudden approximation corresponds to a particle tunneling out of the potential

$$V_s(\eta) = \begin{cases} 0, & \text{if } \eta < \eta_{eq}; \\ 2/(\eta+\eta_{eq})^2 & \text{if } \eta > \eta_{eq}. \end{cases} \tag{11}$$

The singularity in $V(\eta)$ at $\eta = 0$ has been removed and has been replaced by a potential well with width η_{eq} and height $1/2$. This would explain the resonances in the amplitude of $h_s(\eta)$ when $k\eta_{eq} \sim n\pi$ for some integer n, as can be seen explicitly in Eq. (6).

Since $\eta \geq 0$, to apply the WKB approximation define $\eta = e^x$ where $x \in (-\infty, \infty)$. Then, taking $y(x) = e^{x/2}u(x)$,

$$\frac{d^2u}{dx^2} + \left(e^{2x}k^2 - \frac{2e^x}{e^x+2} - \frac{1}{4}\right)u = 0. \tag{12}$$

From the standard WKB connection formulas across the turning point, we find that for $\eta < \eta_T$,

$$h_w(\eta) = \frac{A(k)}{\eta^{1/2}(\eta+2)} \frac{1}{\sqrt{\Gamma(k\eta)}} \exp\left[-\int_{k\eta}^{k\eta_T} \Gamma(s)\frac{ds}{s}\right], \quad \text{and}$$

$$\Gamma(s) = \left(\frac{1}{4} + \frac{2s}{s+2k} - s^2\right)^{1/2}, \tag{13}$$

while for $\eta > \eta_T$,

$$h_w(\eta) = \frac{2A(k)}{\eta^{1/2}(\eta+2)} \frac{1}{\sqrt{K(k\eta)}} \cos\left[\int_{k\eta_T}^{k\eta} K(s)\frac{ds}{s} - \frac{\pi}{4}\right], \quad \text{and}$$

$$K(s) = \left(s^2 - \frac{1}{4} - \frac{2s}{s+2k}\right)^{1/2}. \tag{14}$$

The tunneling amplitude $A(k)$ is

$$A(k) = \sqrt{2\eta_T} \exp\left[\int_0^{k\eta_T} \left(\Gamma(s) - \frac{1}{2}\right) \frac{ds}{s}\right], \tag{15}$$

and η_T is the turning point of the potential in Eq. (12) which is determined by

$$(k\eta_T)^2 - \frac{2\eta_T}{\eta_T + 2} - \frac{1}{4} = 0. \tag{16}$$

One can show that $1/2 \le k\eta_T \le 3/2$. We should also note that when $k\eta$ is large, and when $\eta \gg 2$, the transfer function $T(k)$ of Ref. [7] can be easily obtained from the WKB solution: $T(k) \approx (2/3)k^{3/2}A(k)$.

We shall now compare these three approximate solutions with each other. Taking the small $k\eta << 1$ limit corresponding to modes well outside the horizon, we find that $h_{rm}(\eta) \approx h_s(\eta) \approx 1$ and these solutions are scale invariant. For the WKB solution, on the other hand, we find that $h_w(\eta) \approx H(\eta)$, an η dependent function that is close to unity for all η. This unexpected η-dependence is mainly due to the breakdown of WKB approximation in this limit. Since h satisfies the boundary conditions (2), despite this small deviation from unity the wave amplitude will remain constant when it is outside the horizon.

Next, the asymptotic large $\eta \gg 1$ limit for k fixed at some finite value gives

$$h_{rm}(\eta) \approx -\frac{3}{(k\eta)^2} \cos k\eta, \tag{17}$$

for the MD equation of state while for the sudden approximation,

$$h_s(\eta) \approx \begin{cases} -3\cos k\eta/(k\eta)^2 & \text{if } k \ll 1; \\ 4k\eta_{eq}\cos(k\eta - \pi/2 - k\eta_{eq})/(k\eta)^2 & \text{if } k \gg 1. \end{cases} \tag{18}$$

The WKB approximation gives

$$h_w(\eta) \approx \begin{cases} \frac{8\sqrt{3}}{5}\cos(k\eta - \pi)/(k\eta)^2 & \text{if } k \ll 1; \\ 2\sqrt{\frac{2}{e}}\cos(k\eta - \pi/4)/(k\eta^2) & \text{if } k \gg 1. \end{cases} \tag{19}$$

For small k all three results reduce to Eq. (17), except a slightly different amplitude and an extra phase of π in h_w. This is to be expected. When the wavelength of the GW is much greater than η_{eq}, it does not "see" the RD era and is affected only by the MD universe. Consequently, the SI solution h_{rm} in the MD era works very well for GW of wavelength in the regime $k\eta_{eq} \ll 1$ at late times.

The situation changes dramatically for short-wavelength gravity waves, however. Denoting the amplitudes of $h_{rm}(\eta)$, $h_s(\eta)$ and $h_w(\eta)$ by M_{rm}, M_s and M_w respectively,

$$\frac{M_{rm}}{M_s} \to \frac{3}{4k\eta_{eq}} \quad , \quad \frac{M_{rm}}{M_w} \to \frac{3}{2k}\sqrt{\frac{e}{2}} \quad , \quad \frac{M_s}{M_w} \to 2\eta_{eq}\sqrt{\frac{e}{2}} \tag{20}$$

for $k \gg 1/4$. Since the WKB solution is a closer approximation of the actual history of the universe, we see that the sudden approximation is a very good approximation of the amplitude of the GW. The SI solution in the MD era, on the other hand, severely *underestimates* the amplitude of the GW when $k > 1.75$.

Although the change in the amplitude is most drastic for large k, the relative change in the phase of $h(\eta)$ between the SI solution and the sudden and WKB approximations are present at *all* k and may be significant [13]. In fact, we see that although the amplitude of h_s very closely approximates that of h_w, its phase is k dependent which, more importantly, does not approach any fixed value for large k. Thus, for different values of k the phase difference between h_s and h_w will differ. These phase differences can be seen explicitly by comparing Eqs. (17)-(19) and also in **Fig. 1**. Here we have also plotted the numerical solution $h_n(\eta)$ to Eq. (8).

In our sudden approximation we could have also approximated the equation of state with an a which has a discontinuous first derivative; namely $da/d\eta = 2a/\eta$ for $\eta > \eta_{eq}$. In this case we would have found an h_s which, for $k \gg 4$, underestimates the amplitude of the GW by a factor of 4. Its phase difference, however, approaches a constant value of $\pi/4$ at large k.

We shall now use these results to calculate the anisotropy of CMB induced by the SI spectrum of GW as well as its spectral energy density. From the Sachs-Wolfe effect, the

formula for the power spectrum C_l is given in, for example, [5]. Integrating from the decoupling time $\eta_{dec} \approx 1.54$ for $a_0/a_{dec} = 1100$ and taking the inflation parameter $v = V_0/m_{Pl}^4$, where V_0 and m_{Pl} is the de Sitter vacuum energy and Planck mass respectively, we have numerically evaluated C_l by using both the SI solution h_{rm} and the numerical solution h_n. A comparsion of the two results is given in **Table 1**. As expected, for small l the differences are insignificant since at these values the dominant contribution to C_l come from small k modes, precisely where the SI solutions work well. Due to the difference in the factor of $1/k$ in the amplitudes of the SI and WKB solutions there is a change in C_l for large l, but unlike the results from using transfer functions [7], we find that the SI solution *overestimates* the numerical solution around $l \sim 100$. This, we believe, is due to the phase shift between h_n and h_{rm} which cannot be taken into account by using a simple transfer function.

As for the spectral energy density of the GW

$$\Omega_g \equiv \sum_{\lambda=+,\times} \frac{k}{\rho_c}\frac{d\rho_\lambda}{dk}, \tag{21}$$

which is a measure of the total amount of energy deposited into the experiment at any frequency k. ρ_c is the closure density of the universe. To estimate the amplitude of this quantity, one typically makes use of the following physical argument. Before the GW enters the horizon, it is a pure scale-invariant wave with $h = 1$. After it enters the horizon, on the other hand, the GW behaves effectively as radiation and will scale with a as such. This gives a $\Omega_g \sim 10^{-13}$ for $v \sim 10^{-9}$ [14,5].

We can also calculate Ω_g directly, however. Since

$$\sum_{\lambda=+,\times} k\frac{d\rho_\lambda}{dk} = \frac{v}{3\pi^2 G} k_{\rm phys}^2 |h(\eta)|^2, \tag{22}$$

where G and $k_{\rm phys}$ are the gravitational constant and physical wavenumber respectively, once $h(\eta)$ is known, one can simply evaluate $\Omega_g(\eta)$ at the present time η_0. In fact, using Eq. (7), we find that

$$\Omega_g(\eta_0) = \frac{2v}{9\pi} k^2 |h(\eta_0)|^2 \left(\frac{\eta_0^2 + 2\eta_0}{\eta_0 + 1}\right)^2. \tag{23}$$

Since $\eta_0 \gg 1$ and since any experimentally detectable k is extremely large in our units, we find that by using the SI solutions in Eq. (23),

$$\Omega_g(\eta_0)_{rm} = \frac{2v}{\pi}\frac{1}{(k\eta_0)^2}\cos^2(k\eta_0). \tag{24}$$

Because $k \sim 10^9$, the amplitude of $\Omega_g(\eta_0)_{rm}$ is infinitesimally small, contradicting the above physical argument. If, on the other hand, we use either sudden approximation solution or the WKB solution, then

$$\Omega_g(\eta_0)_s = \frac{32v}{9\pi}\left(\frac{\eta_{eq}}{\eta_0}\right)^2 \cos^2(k\eta_0 - \pi/2 - k\eta_{eq}),$$
$$\Omega_g(\eta_0)_w = \frac{16ve^{-1}}{9\pi}\frac{1}{\eta_0^2}\cos^2(k\eta_0 - \pi/4), \tag{25}$$

and the amplitude of both Ω_g is a constant independent of k.

Since $\Omega_g(\eta)$ is an oscillatory function, we have plotted in **Fig. 2** the *time average*

$$\overline{\Omega}_g \equiv \frac{k}{2\pi}\int_{\eta_0-2\pi/k}^{\eta_0} \Omega_g(\eta)d\eta, \tag{26}$$

of Ω_g over a complete period at the present time verses k for k much larger than the horizon size $k_H = 2\pi/\eta_0$ using the four solutions of (1). It is, moreover, this time average which is measured in experiment and not $\Omega_g(\eta)$. For k near k_H, on the other hand, we have plotted $\Omega_g(\eta_0)$ itself evaluated at the present time η_0 verses k in **Fig. 3** since for these values of k, Ω_g oscillates too slowly for the averaging to be meaningful. Once again we can see from **Fig. 2** that the $\overline{\Omega}_{g_{rm}}$ obtained using h_{rm} severely underestimates the actual $\overline{\Omega}_{g_n}$ obtained using h_n for large k. As expected, $\overline{\Omega}_{g_s}$ and $\overline{\Omega}_{g_w}$ obtained using h_s and h_w are both fairly good approximations of $\overline{\Omega}_{g_n}$. Notice also the dip in $\Omega_g(\eta_0)$ in **Fig**. **3** when $k \approx 0.7k_H$. This sharp decrease in Ω_g at the present horizon size is due the finite size of the universe.

In summary, we have studied the temporal dispersion of a SI spectrum of primordial GW. We have found that the standard SI solution used in the literature severely underestimates the size of the short wavelength, scale-invariant GW at the present time. Although we would thus expect the C_l calculated from the SI solution to always underestimate the actual C_l calculated numerically, this does not turn out to be the case. For $l \sim 100$, it will overestimate the actual C_l and is due to the relative phase shift in the h_n verses h_{rm}. This underestimation by the SI solution also occurs with the calculation of Ω_g, which up to now has been calculated using either indirect physical arguments or by using transfer function methods [7]. This problem does not exist when using either the sudden approximation or the WKB solutions, however, and physically reasonable results for Ω_g can be obtained.

We also see from our solutions that the GW picks up a shift in phase as the universe undergoes the radiation to matter phase transition which is completely overlooked not only in the standard SI solutions, but also in the transfer function method. This phase shift of the GW may conceivably be an invaluable probe of the early universe before the hydrogen recombination. We can imagine future experiments performing interferometry on two incident GW coming from different parts of the sky with their phase shifts being dependent not only on the coincident wavenumber k, but also on the detailed structure of the radiation-matter phase transition. More importantly, it can also serve as a probe of the anisotropy of the early universe as, conceivably, one part of the universe could have undergone the phase transition before another part, thereby producing a relative phase shift. However, the presence of initial randomly distributed phases will complicate this situation, as they should appear in the form of random noise in the measurement of the phase shift. Such experiments would require much greater sensitivity or new techniques than is now possible, of course, and would lie in the far future.

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Captions

Table 1. Table of C_l values calculated with the standard SI solutions, denoted by C_{lrm}, and the numerical solution of Eq. (8), denoted by C_l.

Figure 1. Graphs of $h_{rm}(\eta)$, $h_s(\eta)$, $h_w(\eta)$ and $h_n(\eta)$ to Eq. (8) for $k = 200k_H$ where $k_H = 2\pi/\eta_0$. Notice that among the three approximate solutions, only the WKB solution accurately reproduces the phase of h_n.

Figure 2. Graphs of $\overline{\Omega}_{g_{rm}}$, $\overline{\Omega}_{g_s}$, $\overline{\Omega}_{g_w}$ as well as $\overline{\Omega}_{g_n}$ verses k for large k. Notice the drastic drop in $\overline{\Omega}_{g_{rm}}$.

Figure 3. Graphs of $\Omega_g(\eta_0)_{rm}$, $\Omega_g(\eta_0)_s$, $\Omega_g(\eta_0)_w$ as well as $\Omega_g(\eta_0)_n$ verses k for small k. All three approximate analytical solutions are good approximations of h_n in this regime. The dip in $\Omega_g(\eta_0)$ at $k \approx 0.7k_H$ is actually a point at which Ω_g vanishes and is due to the current finite size of the universe.

Table 1

l	C_{lrm}	C_l	C_l/C_{lrm}
2	7.74	7.76	1.003
20	0.714	0.718	1.006
50	0.255	0.238	0.93
100	0.0611	0.0383	0.63
150	0.00892	0.00223	0.25
200	0.000468	0.00180	3.85
250	0.000479	0.00113	2.36

This figure "fig1-2.png" is available in "png" format from:

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